Matrices and Determinants

Level - 3	Daily Tutorial Sheet-13

156. Applying $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 6x^2a + 2a^3 & 6x^2b + 2b^3 & 6x^2c + 2c^3 \end{vmatrix}$$

Applying $R_3 \to \frac{1}{2}R_3$, then $R_2 \to R_2 + R_3$, we get

$$\Delta = 2\begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ x^3 + 3xa^2 & x^3 + 3xb^2 & x^3 + 3xc^2 \\ 3x^2a + a^3 & 3x^2b + b^3 & 3x^2c + c^3 \end{vmatrix}$$

Applying $R_2 \to (1/x)R_1$ and then $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we obtain

$$\Delta = 2x\begin{vmatrix} x^2 - a^2 & a^2 - b^2 & a^2 - c^2 \\ x^2 + 3a^2 & 3\left(b^2 - a^2\right) & 3\left(c^2 - a^2\right) \\ 3x^2a + a^3 & 3x^2(b-a) + b^3 - a^3 & 3x^2(c-a)c^3 - a^3 \end{vmatrix}$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2 - a^2 & -(a+b) & -(a+c) \\ x^2 + 3a^2 & 3(b+a) & 3(c+a) \\ 3x^2a + a^3 & 3x^2 + b^2 + ab + a^2 & 3x^2 + c^2 + a^2 + ac \end{vmatrix}$$

Applying $R_2 \to R_2 + 3R_1$, and $R_3 \to R_3 + aR_1$, we get

$$\Delta = 2x(b-a)(c-a) \times \begin{vmatrix} x^2 - a^2 & -(a+b) & -(a+c) \\ 4x^2 & 0 & 0 \\ 4x^2a & 3x^2 + b^2 & 3x^2 + c^2 \end{vmatrix}$$

Expanding along R_2, we have

$$\Delta = 8x^3(b-a)(c-a)\left\{(a+b)\left(3x^2 + c^2\right) - (a+c)\left(3x^2 + b^2\right)\right\}$$

$$= 8x^3(b-a)(c-a)\left\{3x^2(b-c) + ac^2 + bc^2 - ab^2 - bc^2\right\}$$

$$= 8x^3(b-a)(c-a)\left\{3x^2(b-c) + bc(c-b) + a\left(c^2 - b^2\right)\right\}$$

$$= 8x^3(b-a)(c-a)(b-c)\left\{3x^2 - (bc + ac + ab)\right\}$$

As a, b, c are distinct, $\Delta = 0$ gives $x = 0$ or $x^2 = 1/3(bc + ca + ab)$, if $ab + bc + ca \le 0$, the only real root is $x = 0$

If $ab + bc + ca > 0$, root are $x = 0, \pm\sqrt{\frac{1}{3}(bc + ca + ab)}$.

157. Applying $C_3 \to C_3 - (C_2 - C_1)$ and $C_4 \to C_4 - (C_1 + C_2)$, we get

$$\Delta = \begin{vmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ a-b & b-c & a+c-2b & 0 \\ x & y & z+x-y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & c & 0 \\ c & b & 0 \\ a-b & b-c & a+c-2b \end{vmatrix} \qquad \text{(expanding along } C_4\text{)}$$

$$= (a+c-2b)\left(ab-c^2\right) \qquad \text{(expanding along } C_3\text{)}$$

$\therefore \quad \Delta = 0$

$\Rightarrow \quad a+c-2b=0 \quad$ or $\quad ab-c^2=0$

Thus, a,b,c are in A.P. or a,c,b are in G.P.

158. We have $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

L. H. S$= \dfrac{1}{(a-b)(b-c)(c-a)} \times \left[\dfrac{f(a)(c-b)}{(x-a)} + \dfrac{f(b)(a-c)}{(x-b)} + \dfrac{f(c)(b-a)}{(x-c)}\right]$

Expanding along C_3)

From R.H.S, by partial fractions, we get

$$\text{R.H.S} = \frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

[since degree of $f(x)$ is less 3]

Then, $A = \left[\dfrac{f(x)}{(x-b)(x-c)}\right]_{x=a} = \dfrac{f(a)}{(a-b)(a-c)}$

Similarly, $B = \dfrac{f(b)}{(b-a)(b-c)}$ and $C = \dfrac{f(c)}{(c-a)(c-b)}$

$\therefore \quad$ R. H. S. $= \dfrac{1}{(a-b)(b-c)(c-a)} \times \left[\dfrac{(c-b)f(a)}{(x-b)} + \dfrac{(a-c)f(b)}{(x-b)} + \dfrac{(b-a)f(c)}{(x-c)}\right]$

Hence, L.H.S = R.H.S

159. $$D = \begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & 2a_p + 3a_{p+m} + 4a_{p+2m} & 4a_p + 9a_{p+m} + 16a_{p+2m} \\ a_q + a_{p+m} + a_{q+2m} & 2a_q + 3a_{q+m} + 4a_{q+2m} & 4a_q + 9a_{q+m} + 16a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & 2a_r + 3a_{r+m} + 4a_{r+2m} & 4a_r + 9a_{r+m} + 16a_{r+2m} \end{vmatrix}$$

$$= \begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & a_{p+m} + 2a_{p+2m} & 5a_{p+m} + 12a_{p+2m} \\ a_q + a_{q+m} + a_{q+2m} & a_{q+m} + 2a_{q+2m} & 5a_{q+m} + 12a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & a_{r+m} + 2a_{r+2m} & 5a_{r+m} + 12a_{r+2m} \end{vmatrix}$$

[Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 4C_1$]

$$= \begin{vmatrix} a_p - q_{p+2m} & a_{p+m} + 2a_{p+2m} & 2a_{p+2m} \\ a_q - a_{q+2m} & a_{q+m} + 2a_{q+2m} & 2a_{q+2m} \\ a_r - a_{r+2m} & a_{r+m} + 2a_{r+2m} & 2a_{r+2m} \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2, C_3 \to C_3 - 5C_2$]

Applying $C_2 \to C_2 - C_3$, then $C_1 \to C_1 + \frac{1}{2}C_3$, then taking 2 common from C_3, we get

$$D = 2\begin{vmatrix} a_p & a_{p+m} & a_{p+2m} \\ a_q & a_{q+m} & a_{q+2m} \\ a_r & a_{r+m} & a_{r+2m} \end{vmatrix} = 2\begin{vmatrix} a_p + a_{p+2m} - 2a_{p+m} & a_{p+m} & a_{p+2m} \\ a_p + a_{q+2m} - 2a_{q+m} & a_{q+m} & a_{q+2m} \\ a_r + a_{r+2m} - 2a_{r+m} & a_{r+m} & a_{r+2m} \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$]

$$= 2\begin{vmatrix} 0 & a_{p+m} & a_{p+2m} \\ 0 & a_{q+m} & a_{q+2m} \\ 0 & a_{r+m} & a_{r+2m} \end{vmatrix}$$ (as a_p, a_{p+m}, a_{p+2m} are in A.P. like others)

$= 0$

160. We know that

$${}^nC_r = \frac{n^{n-1}}{r}C_{r-1}$$

$$\therefore \quad \Delta(n,r) = \begin{vmatrix} \frac{n}{r}\,{}^{n-1}C_{r-1} & \frac{n}{r+1}\,{}^{n-1}C_r & \frac{n}{r+2}\,{}^{n-1}C_{r+1} \\ \frac{n+1}{r}\,{}^{n}C_{r-1} & \frac{n+1}{r+1}\,{}^{n}C_r & \frac{n+1}{r+2}\,{}^{n}C_{r+1} \\ \frac{n+2}{r}\,{}^{n+1}C_{r-1} & \frac{n+2}{r+1}\,{}^{n+1}C_r & \frac{n+2}{r+2}\,{}^{n+1}C_{r+1} \end{vmatrix}$$

$$= \frac{n(n+1)(n+2)}{r(r+1)(r+2)}\Delta(n-1, r-1) \qquad = \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1, r-1) \qquad \text{.... (1)}$$

Repeating the process, we have

$$\Delta(n,r) = \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\Delta(n-2,\ r-2)$$

$$= \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\frac{{}^{n}C_3}{{}^{r}C_3}\cdots\frac{{}^{n-r+3}C_3}{{}^{3}C_3}\Delta(n-r, 0) \qquad \text{..... (2)}$$

Now, $$\Delta(n-r,0) = \begin{vmatrix} {}^{n-r}C_0 & {}^{n-r}C_1 & {}^{n-r}C_2 \\ {}^{n-r+1}C_0 & {}^{n-r+1}C_1 & {}^{n-r+1}C_2 \\ {}^{n-r+2}C_0 & {}^{n-r+2}C_1 & {}^{n-r+2}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 1 & n-r+1 & \frac{1}{2}(n-r+1)(n-r) \\ 1 & n-r+2 & \frac{1}{2}(n-r+2)(n-r+1) \end{vmatrix} = \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 0 & 1 & n-r \\ 0 & 1 & n-r+1 \end{vmatrix}$$

$(R_3 \to R_3 - R_2, R_2 \to R_2 - R_1)$

$= n - r + 1 - n + r = 1$ …… (3)

Hence from (2) and (3), we get

$$\Delta(n,r) = \frac{\left({}^{n+2}C_3\right)\left({}^{n+1}C_3\right)\cdots\left({}^{n-r+3}C_3\right)}{\left({}^{r+2}C_3\right)\left({}^{r+1}C_3\right)\cdots\left({}^{3}C_3\right)}$$

161. The given system of equations will have a non-trivial solution if the determinant of coefficients

$$\Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0 \qquad \text{..... (1)}$$

$\Delta = 0$ is a cubic equation (equation of degree 3) in t, so it has in general three solutions. Let $t_1, t_2,$ and t_3 be the solutions and

$$\Delta = p_0 t^3 + p_1 t^2 + p_2 t + p_3 \qquad \text{...... (2)}$$

Clearly, coefficient of t^3 is $p_0 = -1$. So

$t_1 t_2 t_3 = -\dfrac{p_3}{(-1)} = p_3$ [constant term in the expansion of Δ, i.e., $\Delta(t = 0)$]

Hence, $t_1 t_2 t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$